$$y = \tfrac{1}{2}(-3 \pm \sqrt{13})x.$$

EXAMPLE 10. Let V be $C^{n\times n}$, the space of complex $n \times n$ matrices, and let E^{pq} be the matrix whose only non-zero entry is a 1 in row p and column q. Then the set of all such matrices E^{pq} is orthonormal with respect to the inner product given in Example 3. For

$$(E^{pq}|E^{rs}) = \operatorname{tr}(E^{pq}E^{sr}) = \delta_{qs} \operatorname{tr}(E^{pr}) = \delta_{qs}\delta_{pr}.$$

EXAMPLE 11. Let V be the space of continuous complex-valued (or real-valued) functions on the interval $0 \leq x \leq 1$ with the inner product

$$(f|g) = \int_0^1 f(x)\overline{g(x)}\,dx.$$

Suppose $f_n(x) = \sqrt{2}\cos 2\pi nx$ and that $g_n(x) = \sqrt{2}\sin 2\pi nx$. Then $\{1, f_1, g_1, f_2, g_2, \ldots\}$ is an infinite orthonormal set. In the complex case, we may also form the linear combinations

$$\frac{1}{\sqrt{2}}(f_n + ig_n), \qquad n = 1, 2, \ldots.$$

In this way we get a new orthonormal set S which consists of all functions of the form

$$h_n(x) = e^{2\pi inx}, \qquad n = \pm 1, \pm 2, \ldots.$$

The set S' obtained from S by adjoining the constant function 1 is also orthonormal. We assume here that the reader is familiar with the calculation of the integrals in question.

The orthonormal sets given in the examples above are all linearly independent. We show now that this is necessarily the case.

Theorem 2. *An orthogonal set of non-zero vectors is linearly independent.*

Proof. Let S be a finite or infinite orthogonal set of non-zero vectors in a given inner product space. Suppose $\alpha_1, \alpha_2, \ldots, \alpha_m$ are distinct vectors in S and that

$$\beta = c_1\alpha_1 + c_2\alpha_2 + \cdots + c_m\alpha_m.$$

Then

$$\begin{aligned}(\beta|\alpha_k) &= \Big(\sum_j c_j\alpha_j|\alpha_k\Big)\\ &= \sum_j c_j(\alpha_j|\alpha_k)\\ &= c_k(\alpha_k|\alpha_k).\end{aligned}$$

Since $(\alpha_k|\alpha_k) \neq 0$, it follows that